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NAVAL POSTGRADUATE SCHOOL

Monterey, California

MINIMAL POINT CUBATURES

OF PRECISION SEVEN

FOR SYMMETRIC PLANAR REGIONS

by

Richard Franke

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral A. S. Goodfellow
Superintendent

M. U. Clauser
Provost

Abstract:

A method of constructing 12 point cubature formulas with polynomial precision seven is given for planar regions and weight functions which are symmetric in each variable. If the nodes are real the weights are positive. For any fully symmetric region, or any region which is the product of symmetric intervals, it is shown that infinitely many 12 point formulas exist, and that these formulas use the minimum number of points.

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R. E. Gaskell, Chairman
Department of Mathematics

C. E. Menneken
Dean of Research Administration

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MINIMAL POINT CUBATURES OF PRECISION SEVEN

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Richard Franke*

1. Introduction

We are concerned with determining the minimum number of evaluation points required by certain cubature formulas of the form

$$(2) \qquad \int_R wf \cong \sum_{k=1}^{N} A_k \, f(\nu_k),$$

and exhibiting such minimal point formulas. Here R is a region in n-space and w is a non-negative weight function. A formula (2) which is exact for all polynomials of degree $\le d$, but not for all polynomials of degree $d + 1$ is said to have <u>precision</u> <u>d</u>. We assume the integral exists for all polynomials of degree $\le d$.

For arbitrary regions, Stroud [4,7] has shown that the minimum number of points required by a cubature formula of precision d is $\binom{\left[\frac{d}{2}\right] + n}{n}$. We will be considering the case $n = 2$, $d = 7$, for which the presently known lower bound is the above, 10. Huelsman [2] has recently shown that for fully symmetric regions (i.e., $(x,y) \in R$ implies $(\pm x, \pm y) \in R$ and $(\pm y, \pm x) \in R$) there are no ten point formulas.

Stroud [6,7] has given a characterization of cubature formulas with precision in terms of the evaluation points being zeros of polynomials which

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* Department of Mathematics, Naval Postgraduate School, Monterey, California 93940

have certain orthogonality properties. The important consequence of that characterization which we shall need is the following:

Proposition 3: Let there be given a formula of type (2), with precision 7, for a region in the plane. Suppose that $N < 15$. Then the points ν_k, $k = 1, \ldots, N$ are zeros of $15-N$ linearly independent polynomials of degree 4, each of which is orthogonal, over R with respect to w, to all polynomials of degree ≤ 3.

In the remainder of the paper the following definitions and notation will be used. The function w will be assumed to be symmetric in x and y, i.e., $w(-x,y) = w(x,-y) = w(x,y) \geq 0$. R will be assumed to be symmetric with respect to both axes. We will sometimes speak of a region R and intend this to include the associated weight function w. Q and Q_i will denote polynomials of degree ≤ 3, and P_j will be an orthogonal polynomial of degree 4, i.e., P_j is orthogonal to all Q, over R with respect to w. The five orthogonal polynomials of the form $P^{(m,4-m)} = x^m y^{4-m} + Q_m$, $m = 0, 1, \ldots, 4$ are a basis for the vector space of orthogonal polynomials of degree 4. The integral $\int_R w\, x^p y^q$ will be denoted by I_{pq}. We note that if p or q is odd, $I_{pq} = 0$ and if p and q are both even, $I_{pq} > 0$.

4. Construction of Formulas

The basic idea in constructing our formulas is to determine three linearly independent orthogonal polynomials of degree 4 which have 12 points as common (finite) zeros. These 12 points will be the evaluation points, or nodes, in the formula.

We digress to discuss the properties of the orthogonal polynomials for

our special case. Due to the assumed symmetry of the region and weight function, the basis orthogonal polynomials mentioned earlier have the form

$$
(5) \qquad \begin{aligned}
P^{(4,0)} &= x^4 + a_4x^2 + b_4y^2 + c_4 \\
P^{(3,1)} &= x^3y + a_3yx \\
P^{(2,2)} &= x^2y^2 + a_2x^2 + b_2y^2 + c_2 \\
P^{(1,3)} &= xy^3 + a_1xy \\
P^{(0,4)} &= y^4 + a_0x^2 + b_0y^2 + c_0
\end{aligned}
$$

In the above, $a_1 = -\frac{I_{24}}{I_{22}}$, $a_3 = -\frac{I_{42}}{I_{22}}$, and the remaining coefficients are determined by the equations

$$
\begin{bmatrix} I_{20} & I_{02} & I_{00} \\ I_{40} & I_{22} & I_{20} \\ I_{22} & I_{04} & I_{02} \end{bmatrix} \cdot \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = - \begin{bmatrix} I_{i,4-i} \\ I_{i+2,4-i} \\ I_{i,6-i} \end{bmatrix}, \quad i = 0,2,4
$$

We will consider three orthogonal polynomials

$$
(6) \qquad \begin{aligned}
P_1 &= a_3P^{(1,3)} - a_1P^{(3,1)} = xy(a_3y^2 - a_1x^2) \\
P_2 &= a_3P^{(1,3)} + a_1P^{(3,1)} = xy(a_3y^2 + a_1x^2 + 2a_1a_3) \\
P_3 &= P^{(4,0)} + AP^{(2,2)} + BP^{(0,4)} \\
&= x^4 + Ax^2y^2 + By^4 + Cx^2 + Dy^2 + E,
\end{aligned}
$$

where A and B are to be determined, and then $C = a_4 + Aa_2 + Bb_0$, $D = b_4 + Ab_2 + Bb_0$, and $E = c_4 + Ac_2 + Bc_0$. The reason for considering such polynomials is the following:

<u>Theorem 7</u>: Suppose the orthogonal polynomials (6) have 12 distinct, finite, common zeros. Then these points may be used as nodes in a cubature formula of precision 7 for R and w.

Since the proof of the theorem requires some information about the location of the common zeros, we first investigate whether such polynomials can have 12 common zeros, and if so, how they are distributed. We note that the first two polynomials have xy in common. The other components of the first two have the points $(\pm\alpha, \pm\beta)$ in common, where $\alpha = \sqrt{-a_3}$, $\beta = \sqrt{-a_1}$. Note that since a_1, $a_3 < 0$, these points are real. Now, we require that A and B be chosen so that the points $(\pm\alpha, \pm\beta)$ lie on P_3, i.e., that

$$(8) \qquad P_3(\alpha, \beta) = P^{(4,0)}(\alpha, \beta) + AP^{(2,2)}(\alpha, \beta) + BP^{(0,4)}(\alpha, \beta) = 0.$$

This could fail only if $P^{(2,2)}(\alpha, \beta) = P^{(0,4)}(\alpha, \beta) = 0$, and $P^{(4,0)}(\alpha, \beta) \neq 0$. Therefore we impose the restriction $\mathcal{R}_1$: If $P^{(2,2)}(\alpha, \beta) = P^{(0,4)}(\alpha, \beta) = 0$, then $P^{(4,0)}(\alpha, \beta) = 0$. Condition (8) will generally leave one free parameter, although we won't know in general whether it can be taken as A, or B. Thus we will speak of "the parameter" in P_3.

Now we consider the common zeros of xy and P_3, under condition (8). $P_3(x,0)$ has four zeros, and if $B \neq 0$, $P_3(0,y)$ has four zeros. We want these to be distinct, to give us a total of 12 distinct common zeros. Therefore we now assume that the following restriction is satisfied:

$\mathcal{R}_2$: For some value of the parameter in P_3, $P_3(x,0)$ and $P_3(0,y)$ each have four distinct zeros.

The author knows of no symmetric region for which restrictions $\mathcal{R}_1$ and $\mathcal{R}_2$ are not satisfied, and conjectures that symmetry is sufficient to ensure that they are satisfied. The conditions can be expressed in terms of certain polynomial relations between the I_{pq} being satisfied, or not satisfied. We note, however, that for certain values of the parameter, common zeros may

be repeated, or infinite (e.g., $B = 0$).

The 12 common zeros of the polynomials (6) then have the following form: $(\pm\alpha, \pm\beta)$, $(\pm x_1, 0)$, $(\pm x_2, 0)$, $(0, \pm y_1)$, and $(0, \pm y_2)$, where $x_1^2 \neq x_2^2$, $y_1^2 \neq y_2^2$. We are now prepared to prove Theorem 7.

Proof of Theorem 7: The zeros are symmetric, and we seek a formula such that the weights are also symmetric. The formula will have the form:

$$(9) \quad \int_R wf \cong A_1 \left[f(\alpha, \beta) + f(-\alpha, \beta) + f(\alpha, -\beta) + f(-\alpha, -\beta) \right]$$
$$+ A_2 \left[f(x_1, 0) + f(-x_1, 0) \right] + A_3 \left[f(x_2, 0) + f(-x_2, 0) \right]$$
$$+ A_4 \left[f(0, y_1) + f(0, -y_1) \right] + A_5 \left[f(0, y_2) + f(0, -y_2) \right]$$

Let us solve for $A_1, \ldots, A_5$ by requiring that (9) is exact for the functions $1, x^2, y^2, x^2y^2$, and y^4. The system of equations is:

$$\begin{aligned}
&4A_1 + 2A_2 + 2A_3 + 2A_4 + 2A_5 = I_{00} \\
&4A_1\alpha^2 + 2A_2x_1^2 + 2A_3x_2^2 = I_{20} \\
&4A_1\beta^2 + 2A_4y_1^2 + 2A_5y_2^2 = I_{02} \\
&4A_1\alpha^2\beta^2 = I_{22} \\
&4A_1\beta^4 + 2A_4y_1^4 + 2A_5y_2^4 = I_{04}
\end{aligned}$$

The coefficient matrix is non-singular since all 12 points cannot lie on a polynomial of the form $\mu_0 + \mu_1x^2 + \mu_2y^2 + \mu_3x^2y^2 + \mu_4y^4$.

Now we must show that the resulting formula is exact for the remaining monomials of degree ≤ 7. If p or q is odd, it is exact for x^py^q by symmetry, so that leaves x^4, x^6, x^4y^2, x^2y^4, and y^6. The argument is identical to one in Stroud [5]. We have $\int_R wP_3 = 0$ by orthogonality, and the cubature sum is also zero since all nodes lie on P_3. Because the formula is exact of $1, x^2, y^2, x^2y^2, y^4$ and P_3, it must also be exact for x^4. Considering in

turn $xy(P_1+P_2)$, $xy(P_1-P_2)$, x^2P_3, and y^2P_3, one gets exactness for x^4y^2, x^2y^4, x^6, and y^6 in the same fashion. ∎

The above construction yields a family of cubature formulas of precision 7 for a given region and weight function, the procedure failing only for a finite number of values of the parameter in P_3 (under the assumption of $\mathfrak{R}_1$ and $\mathfrak{R}_2$). For certain values of the parameter the nodes may be complex valued. It would be desirable for the nodes of the formula to be inside R. This is impossible in general, however. For example, for $R = [-1,1] \times [-1,1]$, $w \equiv 1$, no value of the parameter yields a formula with all nodes in the square. For $B = 1$ we obtain a previously known formula due to Mysovskih [3], which has four nodes outside the square. As B is varied, two of these nodes move toward (and into) the square, and two move away from it. There does exist a 12 point formula for the square with all nodes in the interior [8]; however, it does not belong to our family. We will show how to construct it by a similar method in Section 20.

An example to be given in the next section demonstrates that we may not be able to obtain a formula with all the nodes real. However, if the nodes are all real, we have the following result:

Theorem 10: If the nodes of the cubature formula (9) are all real, the weights $A_1, \ldots, A_5$ are all positive.

Proof: We see from the proof of Theorem 7 that $A_1 = \dfrac{I_{22}}{4\alpha^2\beta^2} > 0$. We show that $A_2 > 0$, the positivity of the remaining weights follows by the same method. Consider the polynomial $Q = x\left[\beta^2(x^2-x_2^2) - (\alpha^2-x_2^2)y^2\right]$. All nodes except

$(\pm x_1, 0)$ lie on Q. Since $x_1^2 \neq x_2^2$ and $x_1 \neq 0$, $Q(x_1,0) = Q(-x_1,0) \neq 0$. Q is real valued and non-zero almost everywhere, thus $\int_r wQ^2 = 2A_2 Q^2(x_1,0)$ and $A_2 = \dfrac{\int_r wQ^2}{2Q^2(x_1,0)} > 0$. ∎

We consider an example. Let R be the region bounded by the parabolas $y=\pm(1-x^2)$ with $w \equiv 1$. Then $I_{pq} = \dfrac{4\Gamma(\frac{p+3}{3})\Gamma(q+1)}{(p+1)\Gamma(\frac{p+2q+5}{2})}$ for p and q even. Either A or B may be used as the parameter in P_3; let us speak in terms of B being the parameter. If $B < 0$, some of the nodes are complex. If $B > 0$, the nodes are all real and for a range including approximately the interval (.2, 4.5) the nodes are all in the interior of the region. The value $B = 1$ is perhaps a natural one to consider, and yields a typical self-contained (i.e., all nodes in the region) formula from the family. The formula is given approximately in Table 1.

Points	Weight
$(\pm.52223, \pm.57937)$	.18495
$(\pm.43188, 0)$	.31975
$(\pm.84421, 0)$	.14894
$(0, \pm.41243)$	.33700
$(0, \pm.88401)$	.15775

Table 1: $R = \{(x,y): -1 \le x \le 1,\ |y| \le 1-x^2\}$, $w \equiv 1$

11. Special Case: Fully Symmetric Regions

When R and w are fully symmetric (f.s.) the above details are easier to consider by virtue of the fact the orthogonal polynomials are simplified. In particular, it is true that $I_{pq} = I_{qp}$. Thus we have

$a_4 = b_0$, $b_4 = a_0$, $c_4 = c_0$, $b_2 = a_2$, and $a_3 = a_1$ in the polynomials (5).

Also note that $\alpha = \beta$.

We will require a number of inequalities between the integrals of the monomials. Most of these are obtained by application of the Schwarz inequality. We will list those we need, and prove one to indicate the manner of proof.

$$(12) \qquad I_{pq}^2 < I_{2(p-r),2(q-s)} I_{2r2s}$$

$$(13) \qquad (I_{22} + I_{40})^2 < I_{20} (I_{60} + 3I_{42})$$

$$(14) \qquad I_{42}(I_{40}-I_{22})^2 < (I_{60}-I_{42}) (I_{42}I_{20}-I_{22}^2)$$

$$(15) \qquad 2I_{20}^2 < I_{00}(I_{40}+I_{22})$$

We will prove (14), since it is the most difficult. We give a preliminary result:

$$I_{22}^2 = \left[\int_R wx^2y^2\right]^2 < \left(\int_R wx^2y^2(x^2+y^2)\right) \int_R w \frac{x^2y^2}{x^2+y^2}$$

$$= 2I_{42} \int_R w \frac{x^2y^2}{x^2+y^2}$$

Now we note that for any symmetric integrable function f, we may write $\int_R wf = 4 \int_{R'} w(f(x,y)+f(y,x))$, where $R' = \{ (x,y) \in R: \; 0 \le y \le x \}$. The above inequality then becomes

$$\frac{I_{22}^2}{I_{42}} < 16 \int_{R'} w \frac{x^2y^2}{x^2y^2} \quad . \text{ Now we have}$$

$$(I_{40}-I_{22})^2 = 4\left[\int_{R'} w\,(x^4+y^4-2x^2y^2)\right]^2 = 16\left[\int_{R'} w(x^2-y^2)^2\right]^2$$

$$= 16 \left[\int_{R'} \sqrt{w}\,(x^2-y^2)\,\sqrt{x^2+y^2}\cdot \frac{\sqrt{w}(x^2-y^2)}{\sqrt{x^2+y^2}}\right]^2$$

$$< 16\left(\int_{R'} w(x^2-y^2)^2(x^2+y^2)\right)\int_{R'} \frac{w(x^2-y^2)^2}{x^2+y^2}$$

$$= 16 \int_{R'} w(x^6+y^6-x^4y^2-x^2y^4) . \int_{R'} w(x^2+y^2 - \frac{4x^2y^2}{x^2+y^2})$$

$$= (I_{00}-I_{42}) (I_{20} - 16 \int_{R'} w \frac{x^2y^2}{x^2+y^2})$$

$$< (I_{60}- I_{42}) (I_{20} - \frac{I_{22}^2}{I_{42}}).$$

This is equivalent to (14). We note the strict inequalities appear because the Schwarz inequality is applied to functions such that the square of their quotient is not a constant.

For f.s. regions the orthogonal polynomials (6) can be seen to become:

$$P_1 = a_1 xy(y^2-x^2)$$

(16) $$P_2 = a_1 xy(y^2+x^2-2\beta^2)$$

$$P_3 = x^4+Ax^2y^2+By^4+Cx^2+Dy^2+E,$$

where, as before, $\beta^2 = -a_1 = \frac{I_{42}}{I_{22}}$, A and B satisfy

(17) $$P^{(4,0)}(\beta,\beta) + AP^{(2,2)}(\beta,\beta) + BP^{(0,4)}(\beta,\beta) = 0,$$

and $C = a_4 + Aa_2 + Bb_4$, $D = b_4 + Aa_2 + Ba_4$, and $E = Ac_2 + (1 + B) c_4$.

We first note that restriction $\mathfrak{R}_1$ is satisfied automatically since $P^{(0,4)}$ $P^{(0,4)}(x,y) = P^{(4,0)}(y,x)$, hence $P^{(0,4)}(\beta,\beta) = 0$ implies $P^{(4,0)}(\beta,\beta) = 0$. We consider the common zeros of the polynomials (16) with condition (17) to show that restriction $\mathfrak{R}_2$ is also satisfied.

We have $P_3(x,0) = x^4 + (a_4+Aa_2+Bb_4)x^2 + (Ac_2 + (1+B) c_4)$ where (17) is satisfied. For the zeros to be distinct we need (i) $E = Ac_2 + (1+B) c_4 \neq 0$, and (ii) $C^2-4E = (a_4+Aa_2+Bb_4)^2 -4 (Ac_2 + (1+B) c_4) \neq 0$.

Case (i). Assume $Ac_2 + (1+B) c_4 \equiv 0$ for all values of the parameter in P_3.

Condition (17) then yields $c_4 P^{(2,2)}(\beta,\beta) - c_2 P^{(4,0)}(\beta,\beta) = 0$. We first demonstrate that c_2 and c_4 cannot be zero simultaneously. We have

$$\Delta c_4 = I_{40}(I_{40} + I_{22}) - I_{20}(I_{00} + I_{42}) \text{ and}$$

$$\Delta c_2 = I_{22}(I_{40} + I_{22}) - 2I_{20}I_{42}, \text{ where}$$

$$\Delta = 2I_{20}^2 - I_{00}(I_{40} + I_{22}).$$

$$\text{Then } \Delta(c_4 + c_2) = I_{22}^2 + 2I_{22}I_{40} + I_{40}^2 - I_{20}(I_{60} + 3I_{42})$$

$$= (I_{22} + I_{40})^2 - I_{20}(I_{60} + 3I_{42}) < 0$$

by (13).

We now show that $c_4 P^{(2,2)}(\beta,\beta) - c_2 P^{(4,0)}(\beta,\beta)$ cannot be zero. If it were, there would exist a non-trivial orthogonal polynomial of the form $P_4 = \mu_0 x^4 + \mu_1 x^2 y^2 + \mu_2 x^2 + \mu_3 y^2$ which is zero at (β,β). Thus the system of homogeneous equations $P_4(\beta,\beta) = 0$, $\int_R w P_4 x^p y^q = 0$ for $(p,q) = (0,0), (2,0), (0,2)$ must have a singular coefficient matrix. The determinant of the coefficient matrix is

$$\begin{vmatrix} \beta^4 & \beta^4 & \beta^2 & \beta^2 \\ I_{40} & I_{22} & I_{20} & I_{20} \\ I_{60} & I_{42} & I_{40} & I_{22} \\ I_{42} & I_{42} & I_{22} & I_{40} \end{vmatrix},$$

which expands to

$$\frac{I_{42}}{I_{22}^2}(I_{22} - I_{40})\left[I_{42}(I_{40} - I_{22})^2 + (I_{60} - I_{42})(I_{22}^2 - I_{20}I_{42})\right].$$

Then by (12) and (14), the determinant is positive.

Case (ii). Assume that

(18) $(a_4 + Aa_2 + Bb_4)^2 - 4 [Ac_2 + (1+B) c_4] \equiv 0$, when condition (17) is satisfied. $P^{(2,2)}(\beta,\beta) = 0$ requires $B = -1$, and A as the parameter. In that instance we have $(a_4 - b_4 + Aa_2)^2 - 4Ac_2 \equiv 0$ which would require that $a_2 = c_2 = (a_4 - b_4) = 0$. But $a_2 = c_2 = 0$ implies $P^{(2,2)} = x^2y^2$, clearly an impossibility. Thus we may suppose that $P^{(2,2)}(\beta,\beta) \neq 0$. (Note: It is possible for $P^{(2,2)}(\beta,\beta)$ to be zero, but not, of course, under condition (18)).

Now we may write $A = -(1+B)\dfrac{P^{(4,0)}(\beta,\beta)}{P^{(2,2)}(\beta,\beta)} = (1+B)\mu$.

Then (18) becomes

$$[a_4 + (1+B)\mu a_2 + Bb_4]^2 - 4 [(1+B)\mu c_2 + (1+B) c_4] \equiv 0.$$

We write this as a quadratic in 1+B, obtaining

$$(1+B)^2 [\mu a_2 + b_4]^2 + (1+B) [2(a_4-b_4)(\mu a_2+b_4) - r(\mu c_2 + c_4)] + (a_4-b_4)^2 \equiv 0.$$

Thus, we must have $\mu c_2 + c_4 = 0$. But this is equivalent to $c_4P^{(2,2)}(\beta,\beta) - c_2P^{(4,0)}(\beta,\beta) = 0$, which was shown to be impossible in case (i).

For the zeros of $P_3(0,y)$ to be distinct and finite we must have

(iii) $E \neq 0$

(iv) $D^2 - 4BE \neq 0$

(v) $B \neq 0$.

Case (iii) is case (i), and case (iv) is similar to case (ii). Since $P^{(0,4)}(x,y) = P^{(4,0)}(y,x)$, and in particular, $P^{(0,4)}(\beta,\beta) = P^{(4,0)}(\beta,\beta)$, we can always take $B \neq 0$. If $P^{(2,2)}(\beta,\beta) = 0$, we must take $B = -1$, however, assuming that $P^{(4,0)}(\beta,\beta) \neq 0$. In any case, restriction $\mathcal{R}_2$ is satisfied.

We have now completed the proof of the following theorem.

<u>Theorem 19</u>: For all but a finite number of values of the parameter in P_3,

there is a corresponding 12 point cubature formula of precision 7 for any fully symmetric region R and weight function w.

For f.s. regions it is desirable to have a f.s. formula. This would be obtained by taking $B = 1$, if that is possible. This construction would fail to yield a f.s. formula if any one of conditions (ii) - (iv) fail for $B = 1$, or if $P^{(2,2)}(\beta,\beta) = 0$. We give an example where the latter occurs.

Consider the family of f.s. polygonal regions with vertices at $(\pm 1,\pm 1)$, $(\pm t,0)$, and $(0,\pm t)$, where $t > 0$ is a parameter. Let $w \equiv 1$ on R. The I_{pq} are polynomials in t, thus $P^{(2,2)}(\beta,\beta)$ is a rational function of t. The numerator of $P^{(2,2)}(\beta,\beta)$ (t) has a zero $t_0 \cong .60584$. For $t = t_0$ then, none of the formulas given by our construction is fully symmetric, as $B = -1$ for all of them. A representative formula is given in Table 2, and corresponds to $A = 1$. It is easy to see that all of the formulas we obtain for this region involve complex nodes.

Points	Weight
$(\pm.74553,\ \pm.74553)$	.17834
$(\pm.32252, 0)$	.79365
$(\pm.90057, 0)$	.02390
$(0, \pm.64198)$	.17769
$(0, \pm.45243i)$	-.14023

Table 2: $R = \left\{ (\pm x,\pm y),\ (\pm y,\pm x): 0 \le y \le x \le t_0 + y(1-t_0),\ t_0 \cong .60584 \right\}$, $w \equiv 1$.

20. Alternate Construction: Fully Symmetric Regions

We now consider an alternate construction for f.s. regions which yields f.s. formulas, when it is successful. The spirit of the method is identical

to that of the previous method. We consider the following orthogonal polynomials:

(21)
$$
\begin{aligned}
P_1 &= P^{(1,3)} - P^{(3,1)} = xy(y^2-x^2) \\
P_2 &= P^{(4,0)} - P^{(0,4)} = (x^2-y^2)\ (x^2+y^2-\gamma^2) \\
P_3 &= P^{(4,0)} + AP^{(2,2)} + P^{(0,4)} \\
&= x^4 + Ax^2y^2 + y^4 + C(x^2 + y^2) + E,
\end{aligned}
$$

where $\gamma^2 = b_4 - a_4$, $C = a_4 + b_4 + Aa_2$, $E = 2c_4 + Ac_2$, and A is selected so that

(22)
$$P_3(\gamma, 0) = 0.$$

We then have $A = \dfrac{P^{(4,0)}(\gamma,0) + P^{(0,4)}(\gamma,0)}{P^{(2,2)}(\gamma,0)}$, hence we must have $P^{(2,2)}(\gamma,0) \neq 0$.

If we assume condition (22) can be satisfied, the common zeros of the polynomials (21) are $(\pm\gamma, 0)$, $(0, \pm\gamma)$, $(\pm\delta, \pm\delta)$, and $(\pm\tau, \pm\tau)$, where δ and τ are zeros of $P_3(x,x)$. If these fail to be distinct, or if $P_3(x,x)$ has degree 2, the construction fails.

We could, of course, allow another parameter in P_3, as we did previously. This would probably ensure the existence of formulas of this type; however, our goal here was to attempt to construct a f.s. formula if the previous construction failed for B = 1.

This formula for the region given previously ($t = t_0 \cong .60584$) is given in Table 3. It is not self-contained.

Points	Weight
$(\pm.34308, \pm.34308)$	.39246
$(\pm.77704, \pm.77704)$	.13621
$(\pm.74916, 0)$	.07716
$(0, \pm.74916)$	.07716

Table 3: $R = \left\{ (\pm x, \pm y), (\pm y, \pm x): 0 \le y \le x \le t_0 + y(1-t_0), t_0 \cong .60584 \right\}$, $w \equiv 1$

We noted previously that our original construction failed to give a self-contained formula for the square. The alternate construction yields a formula previously given by Tyler [8]. We hasten to note that the alternate construction is not applicable to arbitrary symmetric regions.

We point out an example of failure of the alternate procedure due to $P_3(x,x)$ being of degree 2. Consider again the family of f.s. polygonal regions with vertices at $(\pm 1, \pm 1)$, $(\pm t, 0)$, $(0, \pm t)$. For $t = t_2 \cong 3.3$, we obtain $A = -2$, and thus $P_3(x,x)$ has but two distinct zeros, not the four required. Four of the common zeros of $y^2 - x^2$ and P_3 are at infinity.

23. Minimal Point Formulas: Fully Symmetric Regions

Theorem 24: Let R and w be fully symmetric. Then any cubature formula of precision 7 for R and w uses at least 12 nodes.

Proof: Proposition 4 assures us that any such formula must use at least 10 points. Huelsman [2] shows that there are no 10 point formulas. Thus we need only to show that no 11 point formula exists. We assume the contrary: There exists a f.s. region for which an 11 point formula exists. Let the nodes be denoted by ν_k, $k = 1, \ldots, 11$.

Now the ν_k all lie on four linearly independent orthogonal polynomials of degree 4. Thus they lie on some non-trivial linear combination of any two of the basis polynomials (5). Hence, we deduce that the ν_k all lie on

$$(25) \qquad xy\,[\lambda_1 x^2 + \mu_1 y^2 + a_1\,(\lambda_1 + \mu_1)\,]$$

Thus each of the ν_k lie on either the conic $\lambda_1 x^2 + \mu_1 y^2 + a_1\,(\lambda_1 + \mu_1)$, or one of the axes. Now the ν_k must also lie on the polynomials

$$(26) \qquad \begin{array}{l} \lambda_2 P^{(4,0)} + \mu_2 P^{(2,2)} \\ \lambda_3 P^{(0,4)} + \mu_3 P^{(2,2)}. \end{array}$$

Note that these two need not be linearly independent, if all the ν_k lie on $P^{(2,2)}$. The polynomials (26) have at most two distinct common zeros on each of the x and y axes, for a total of four.

The remaining ν_k must then lie on the conic $\lambda_1 x^2 + \mu_1 y^2 + a_1(\lambda_1 + \mu_1)$. Two possibilities present themselves: (I) The ν_k all lie on one of $P^{(1,3)}$ or $P^{(3,1)}$, (II) the ν_k do not all lie on either of $P^{(1,3)}$ or $P^{(3,1)}$.

We consider case (I). Suppose at least seven of the ν_k lie on $x^2 + a_1$. Now $x^2 + a_1$ and $\lambda_2 P^{(4,0)} + \mu_2 P^{(2,2)}$ have at most four finite common zeros, and this is not enough. If seven of the ν_k lie on $y^2 + a_1$, the argument is dual.

For case (II), we have both λ_1 and μ_1 non-zero. The ν_k lie on the following orthogonal polynomials:

$$\begin{aligned} P_1 &= xy\,[\lambda_1 x^2 + \mu_1 y^2 + a_1(\lambda_1 + \mu_1)\,] \\ P_2 &= \lambda_2 P^{(4,0)} + \mu_2 P^{(2,2)} \\ P_3 &= \lambda_3 P^{(0,4)} + \mu_3 P^{(2,2)} \\ P_4 &= \lambda_4 P^{(4,0)} + \mu_4 P^{(3,1)} \end{aligned}$$

Now we note that the common zeros of P_1 and P_2 appear in symmetric pairs. Hence, they must have eight distinct common zeros of the form $(\pm x_1, \pm y_1)$, $(\pm x_2, \pm y_2)$, where none of these points lie on either axis, and at least seven are nodes in the formula. At least three of the points $(\pm x_1, \pm y_1)$ lie on P_4. Since $P^{(4,0)}$ is an even function of both x and y, and $P^{(3,1)}$ is an odd function of both x and y, we deduce that $\lambda_4 P^{(4,0)}(x_1, y_1) \pm \mu_4 P^{(3,1)}(x_1, y_1) = 0$. Thus $\lambda_4 P^{(4,0)}(x_1, y_1) = \mu_4 P^{(3,1)}(x_1, y_1) = 0$. Similarly we find that $\lambda_4 P^{(4,0)}(x_2, y_2) = \mu_4 P^{(3,1)}(x_2, y_2) = 0$. Since not all eight of the points lie on $P^{(3,1)}$, we must have $\mu_4 = 0$.

We now have established that the four linearly independent orthogonal polynomials are

$$P_1 = P^{(3,1)} + \mu_1 P^{(1,3)}$$
$$P_2 = P^{(2,2)}$$
$$P_3 = P^{(0,4)}$$
$$P_4 = P^{(4,0)} \quad .$$

Since the ν_k lie on $P_4 = P^{(4,0)}$ and $P_3 = P^{(0,4)}$, they are a subset of the common zeros of those two polynomials. The common zeros of $P^{(4,0)}$ and $P^{(0,4)}$ also lie on $P^{(4,0)} - P^{(0,4)} = (x^2 - y^2)(x^2 + y^2 - \gamma^2)$. Thus eight zeros lie on $x^2 - y^2$ and eight on $x^2 + y^2 - \gamma^2$. Since we have four zeros on the axes, it is clear that they must be $(0, \pm\gamma)$, $(\pm\gamma, 0)$, and these are each of multiplicity two as common zeros of $P^{(4,0)}$ and $P^{(0,4)}$. Now $P_1 = xy\,[x^2 + \mu_1 y^2 + (1 + \mu_1) a_1]$ is zero at the above points. It is easily seen that the eight zeros on $x^2 - y^2$ can lie on P_1 only if $\mu_1 = -1$.

Then for all common zeros not on the axes, the ordinate and abscissa have the same absolute value. Thus the zeros of $P^{(4,0)}(x,x)$ must be the same as those of $P^{(2,2)}(x,x)$. We have $P^{(4,0)}(x,x) = x^4 + (a_4+b_4)x^2 + c_4$, $P^{(2,2)}(x,x) = x^4 + 2a_2x^2 + c_2$. Thus $a_4 + b_4 = 2a_2$ and $c_4 = c_2$. Considering the common zeros on the axes gives us $P^{(2,2)}(x,0) = a_2x^2 + c_2 = 0$, or $x^2 = -\frac{c_2}{a_2}$. Then $P^{(0,4)}\left(\sqrt{-\frac{c_2}{a_2}}, 0\right) = -\frac{c_2}{a_2} b_4 + c_4 = 0$, or $c_2b_4 = c_4a_2$. The previous condition $c_2 = c_4$ implies $a_2 = b_4$, since we must have $c_2 \neq 0$. $a_4 + b_4 = 2a_2$ then gives $a_4 = b_4 = a_2$. The condition $P^{(4,0)}\left(\sqrt{-\frac{c_2}{a_2}}, 0\right) = 0$ gives us $0 = \frac{1}{a_2^2}\left[c_2^2 - a_2a_4c_2 + c_4a_2^2\right] = \frac{c_2}{a_2^2}\left[c_2 - a_2(a_4 - b_4)\right]$. Since $c_2 \neq 0$, we have $c_2 = a_2(a_4 - b_4)$, but $a_4 = b_4$ gives the contradiction $c_2 = 0$. Thus $\mu_1 \neq -1$ either, and we have completed the proof.

27. Special Case: Symmetric Product Regions

When $R = [-a,a] \times [-b,b]$ and $w(x,y) = u(x)v(y)$ where u and v are even functions we can obtain the same results as for f.s. regions. We refer to these regions as symmetric product (s.p.) regions.

For s.p. regions the basis orthogonal polynomials, $P^{(m,4-m)}$, $m=0,\ldots,4$, are products of orthogonal polynomials in one variable. Thus if $L^{(k)}(x)$ is the monic polynomial which is orthogonal to all polynomials of degree $<k$, over $[-a,a]$ with respect to $u(x)$, and if $M^{(k)}(y)$ is the corresponding polynomial for $[-b,b]$ and $v(y)$, then $P^{(m,4-m)} = L^{(m)}(x)M^{(4-m)}(y)$. It can be shown that the orthogonal polynomials $P^{(m,4-m)}$ are of the form

$$P^{(4,0)} = (x^2-x_1^2)(x^2-x_4^2)$$

$$P^{(3,1)} = xy\,(x^2-x_3^2)$$

$$P^{(2,2)} = (x^2-x_2^2)\ (y^2-y_2^2)$$
$$P^{(1,3)} = xy(y^2-y_3^2)$$
$$P^{(0,4)} = (y^2-y_1^2)\ (y^2-y_4^2)\ .$$

It is easy to show that the inequalities

$$0 < x_1^2 < x_2^2 < x_3^2 < x_4^2 < a^2 \quad \text{and}$$
$$0 < y_1^2 < y_2^2 < y_3^2 < y_4^2 < b^2 \quad \text{are valid.}$$

In the case R is a s.p. region, the orthogonal polynomials corresponding to (6) are

$$P_1 = -xy\ (x_3^2y^2 - y_3^2x^2)$$

(28) $$P_2 = -xy\ (x_3^2y^2 + y_3^2x^2 - 2\,x_3^2y_3^2)$$

$$P_3 = x^4 + Ax^2y^2 + By^4$$
$$- (x_1^2 + x_4^2 - Ay_2^2)\,x^2 - (By_1^2 + By_4^2 + Ax_2^2)\,y^2$$
$$+ x_1^2x_4^2 + Ax_2^2y_2^2 + By_1^2y_4^2$$

with A and B chosen so as to satisfy

(29) $$P_3(\alpha,\beta) = (x_3^2 - x_1^2)(x_3^2 - x_4^2) + A\,(x_3^2 - x_2^2)(y_3^2 - y_2^2)$$
$$+ B(y_3^2 - y_1^2)\ (x_3^2 - x_1^2) = 0$$

where $(\alpha,\beta) = (x_3, y_3)$.

The inequalities between the x_i and y_i assure we have restriction $\Re_1$ satisfied, with B as a parameter. Then

$$A = -\frac{(x_3^2 - x_1^2)\ (x_3^2 - x_4^2) + B(y_3^2 - y_1^2)\ (y_3^2 - y_4^2)}{(x_3^2 - x_2^2)\ (y_3^2 - y_2^2)} \quad .$$

We now consider the zeros of $P_3(x,0)$ with B as a parameter.

$$P_3(x,0) = x^4 - (x_1^2 + x_4^2 + Ay_2^2) + (x_1^2x_4^2 + Ax_2^2y_2^2 + By_1^2y_4^2),$$

so if $\quad x_1^2x_4^2 + Ax_2^2y_2^2 + By_1^2y_4^2 \neq 0$ and

$$(x_1^2 + x_4^2 + Ay_2^2)^2 - 4(x_1^2x_4^2 + Ax_2^2 + By_1^2y_2^2) \neq 0$$

for some value of B, the zeros will be distinct. Substituting A in terms of B in the first gives us

$$\frac{1}{(x_3^2-x_2^2)(y_3^2-y_2^2)}\left[(x_3^2 - x_2^2)(y_3^2 - y_2^2)\, x_1^2x_4^2 - (x_3^2 - x_1^2)(x_3^2 - x_4^2)\, x_2^2y_2^2\right.$$
$$\left. - B(y_3^2 - y_1^2)(y_3^2 - y_4^2)\, x_2^2y_2^2 + By_1^2y_4^2\,(x_3^2 - x_2^2)(y_3^2 - y_2^2)\right]$$
$$= \frac{(x_3^2 - x_2^2)(y_3^2-y_2^2)x_1^2x_4^2 - (x_3^2 - x_1^2)(x_3^2 - x_4^2)\, x_2^2y_2^2}{(x_3^2 - x_2^2)(y_3^2 - y_2^2)}$$
$$+ B\,\frac{- (y_3^2 - y_1^2)(y_3^2 - y_4^2)\, x_2^2y_2^2 + (x_3^2 - x_2^2)(y_3^2 - y_2^2)\, y_1^2y_4^2}{(x_3^2 - x_2^2)(y_3^2 - y_2^2)}$$

Inspection of the above in the light of the inequalities between the x_i and y_i shows us that the coefficient of B is positive, hence the expression is non-zero for all but one value of B.

Considering in a similar way the second expression, one finds that the coefficient of B^2 is

$$\left[\frac{(y_3^2 - y_1^2)(y_3^2 - y_4^2)}{(x_3^2 - x_2^2)(y_3^2 - y_2^2)}\right]^2 \neq 0,$$ hence the expression is zero for at most two values of B.

The polynomial $P_3(0,y)$ must also have four distinct zeros, and similar consideration gives the result for all but a finite number of values of B. Thus we have proved the theorem now given.

Theorem 30: For all but a finite number of values of B, there is a corresponding 12 point cubature formula of precision 7 for any symmetric product region R and weight function w.

We consider an example. Let $R = (-\infty,\infty) \times [-1,1]$ and $w(x,y) = e^{-x^2}$.

Then

$$I_{pq} = \frac{p!\sqrt{\pi}}{(q+1)2^{p-1}(p/2)!}$$

for p and q both even. For $B > 7.5$ the formulas are self-contained. A typical one, for $B = 10$, is given approximately in Table 4.

Point	Weight
$(\pm 1.22475, \pm .77460)$	.16412
$(\pm .75942, 0)$	.54525
$(\pm 2.27056, 0)$	.01541
$(0, \pm .55770)$	.69894
$(0, \pm .97772)$	.18462

Table 4: $R = (-\infty,\infty) \times [-1,1]$, $w(x,y) = e^{-x^2}$

31. Minimal Point Formulas: Symmetric Product Regions

Theorem 32: Let R be a symmetric product region. Then any cubature formula of precision 7 for R and w uses at least 12 points.

Proof: The proof we are going to give here has a slightly different flavor than that given for f.s. regions. We do so because it is felt that the use of algebraic geometry may be indicative of the type of proof which may be necessary for more complex regions. We note that the only common zeros that $L^{(k)}(x)$, $k = 1,2,3,4$ can have are at the origin, and similarly for $M^{(k)}(y)$, $k = 1,2,3,4$. We need a preliminary result.

Proposition 33: Let $\ell(x)$ be a linear component of $L^{(m)}(x)$ which is not x. Then the only orthogonal polynomials of degree 4 (over $[-a,a] \times [-b,b]$ with respect to $u(x)v(y)$) which have $\ell(x)$ as a component are multiples of $L^{(m)}(x)M^{(4-m)}(y)$.

Proof: Suppose that $\ell(x)Q(x,y)$ is an orthogonal of degree 4, and that $\ell(x^*)=0$. Then $(x-x^*)Q(x,y) = \sum_{k=0}^{4} \lambda_k L^{(k)}(x)M^{(4-k)}(y)$, so $\sum_{k=0}^{4} \lambda_k L^{(k)}(x^*)M^{(4-k)}(y) \equiv 0$. This is possible only if $\lambda_k L^{(k)}(x^*) = 0$ for $k = 0,1,\ldots,4$. Thus $\lambda_k = 0$ for $k \neq m$, and the proposition is established. We note the identical result holds in y.

We have noted previously that no cubature formula of precision 7 can use fewer than ten points. For s.p. regions it is clear none can use ten, since $L^{(2)}(x)M^{(2)}(y)$ and $L^{(4)}(x)$ have only eight common finite zeros. Thus we need only to show no s.p. region exists which has an 11 point formula. Assume the contrary, and let the nodes be ν_k, $k = 1,\ldots,11$.

Proposition 34: The ν_k do not all lie on $L^{(4)}(x)$, $L^{(2)}(x)M^{(2)}(y)$, or $M^{(4)}(y)$.

Proof: The argument is similar for all three; we consider $L^{(4)}(x)$. At least one linear component of $L^{(4)}(x)$ must have ≥ 3 of the ν_k on it. Say $L^{(4)}(x) = \ell(x)Q(x)$ where ≥ 3 of the ν_k lie on $\ell(x)$. Because any polynomial of degree four on which all the ν_k lie must be an orthogonal polynomial, and because there are at least two linearly independent $Q(x,y)$ passing through the ≤ 8 ν_k not on $\ell(x)$, we have two linearly independent orthogonal polynomials of the form $\ell(x)Q(x,y)$. This contradicts Proposition 33.

Now we are assured that the ν_k lie on four linearly independent orthogonal polynomials of the form

$$P_1 = L^{(4)}(x) + \lambda_1 L^{(2)}(x)M^{(2)}(y)$$
$$P_2 = L^{(3)}(x)M^{(1)}(y) + \lambda_2 L^{(2)}(x)M^{(2)}(y)$$
$$P_3 = L^{(1)}(x)M^{(3)}(y) + \lambda_3 L^{(2)}(x)M^{(2)}(y)$$
$$P_4 = M^{(4)}(y) + \lambda_4 L^{(2)}(x)M^{(2)}(y),$$

where λ_1 and λ_4 are non-zero.

Suppose the ν_k all lie on $L^{(3)}(x)M^{(1)}(y) = xy(x^2-x_3^2)$. We know that not more than two of the ν_k could lie on either of $x \pm x_3$, since if ≥ 3 were on $x + x_3$ (say), there are ≥ 2 linearly independent Q such that each of the $(x + x_3)Q$ is an orthogonal polynomial. This is not possible by Proposition 33. Thus there are at most four of the ν_k not on xy. Then we see that there are ≥ 2 linearly independent orthogonal polynomials of the form xyQ_2, where Q_2 is of degree 2, on which all the ν_k lie. This follows because there are ≥ 2 linearly independent Q_2 passing through four points. We must have

$$xyQ_2 = \sum_{m=0}^{4} \mu_m L^{(m)}(x)M^{(m)}(y). \quad \text{If } x = 0,$$

we have $$\mu_0 M^{(4)}(y) + \mu_2 L^{(2)}(0)M^{(2)}(y) + \mu_4 L^{(4)}(0) \equiv 0.$$

This is possible only if $\mu_0 = \mu_2 = \mu_4 = 0$. Thus the two linearly independent xyQ_2 are $L^{(3)}(x)M^{(1)}(y)$ and $L^{(1)}(x)M^{(3)}(y)$, so if the ν_k all lie on one of $L^{(3)}(x)M^{(1)}(y)$ or $L^{(1)}(x)M^{(3)}(y)$, they lie on both. We must have nodes off the axes, and these can only be $(\pm x_3, \pm y_3)$. The polynomials P_1 and P_4 can have at most two zeros on the y and x axes, respectively. Thus we cannot obtain 11 common zeros for P_1, P_2, P_3, and P_4.

The remaining possibility is that both λ_2 and λ_3 are non-zero. Then the ν_k all lie on $\lambda_3 P_2 - \lambda_2 P_3 = xy\ [\lambda_3(x^2-x_3^2) - \lambda_2\ (y^2-y_3^2)\ ]$. If ν_j lies on the x-axis, we have $\nu_j = (\pm x_2, 0)$, since $\pm x_2$ are the only zeros of $P_2(x,0)$. Likewise $(0,\pm y_2)$ are the only zeros of P_3 on the y-axis. But then

$P_1(\pm x_2, 0) = L^{(4)}(\pm x_2) + \lambda_1 L^{(2)}(\pm x_2) M^{(2)}(0) = L^{(4)}(x_2) \neq 0$. Similarly $P_4(0,\pm y_2) = M^{(4)}(y_2) \neq 0$. Hence there can be no nodes on the axes. But then they must all lie on $\lambda_3(x^2 - x_3^2) - \lambda_2(y^2 - y_3^2)$. This is impossible since not all of the nodes of the formula can lie on a polynomial of degree ≤ 3.

■

35. Conclusions

It is known that the minimum number of points required by a cubature formula of specified precision depends on the region. We have exhibited 12 point formulas for f.s. and s.p. regions, which have precision 7, and shown this to be the minimum number of points possible. This answers affirmatively a conjecture by Stroud [7, Section 3.16] that certain known 12 point cubatures of precision 7 are minimal point rules. In an earlier section the author conjectured the result holds for arbitrary symmetric regions as well.

The author knows of no region for which a 10 point or 11 point formula exists, and it would be interesting to know if there is one. Likewise, we might ask: Is there a planar region for which the minimal number of points for a formula of precision 7 is greater than 12? The author conjectures that the triangular region may be a candidate. See [1] for some computations on this problem. A proof similar to the above does not seem likely, however.

The extension of the above approach to other regions, especially in more dimensions, and other degrees of precision, does not appear to be straightforward. The present analysis was made possible by the rather special circumstance regarding the number of polynomials of degree $\left[\frac{d}{2}\right]+1$ of which the nodes must be common zeros. For example, for fully symmetric planar regions, the author would conjecture that the minimum number of points for formulas of precision 9 would be 20. However, we could be sure the nodes would lie on at most one polynomial of degree 5. The author has found a 20 point formula for the square, the nodes actually being common zeros of two orthogonal polynomials of degree 5 [1].

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13. ABSTRACT

A method of constructing 12 point cubature formulas with polynomial precision seven is given for planar regions and weight functions which are symmetric in each variable. If the nodes are real the weights are positive. For any fully symmetric region, or any region which is the product of symmetric intervals, it is shown that infinitely many 12 point formulas exist, and that these formulas use the minimum number of points.

14 KEY WORDS	LINK A		LINK B		LINK C	
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